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A complete $L(2, 1)$ span characterization for small trees

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Abstract

An $L(2, 1)$ labeling of a graph G is a vertex labeling such that any pair of vertices v_i and v_j must have labels at least 2 apart if $d(v_i, v_j) = 1$ and labels at least 1 apart if $d(v_i, v_j) = 2$. The span of an $L(2, 1)$ labeling f on a graph G is the maximum $f(u)$ for all $u \in V(G)$. The $L(2, 1)$ span of a graph G is the minimum span of all $L(2, 1)$ labelings on G. The $L(2, 1)$ labeling on trees has been extensively studied in recent years. In this paper we present a complete characterization of the $L(2, 1)$ span of trees up to twenty vertices.

Keywords: Channel assignment problem; $L(2, 1)$ labeling; Chang–Kuo algorithm; Forbidden subtree characterization

1. Introduction

The channel assignment problem, introduced by Hale and later modified by Roberts [1], describes the assignment of frequencies to transmitters so as to decrease interference. Griggs and Yeh studied a variation which stipulates that labels also depend on the distance from the corresponding vertex to other nearby vertices in the same graph [2].

Formally an *$L(h, k)$-labeling* of a graph G is a nonnegative integer labeling of the vertices where adjacent vertices differ in label by at least h, and vertices that are at distance two from each other differ in label by at least k. The *span of an $L(h, k)$ labeling f* on a graph G is the maximum $f(u)$ for all $u \in V(G)$. The *$L(h, k)$ span of a graph G*, denoted $\lambda_{h,k}(G)$, is the minimum span of all $L(h, k)$ labelings on G. An $L(h, k)$ labeling f on G whose span is equal to the span of G is called a *span labeling* of G.

The $L(2, 1)$-labeling problem on trees has been studied extensively. Griggs and Yeh showed in [2] that $\lambda_{2,1}(T) \in \{\Delta(T) + 1, \Delta(T) + 2\}$ for all trees T, and further conjectured that the problem of recognizing the two classes of trees is NP-hard. However, Chang and Kuo [3] have since provided a polynomial-time algorithm that can decide whether or not the $L(2, 1)$-span for a tree T is $\Delta(T) + 1$.

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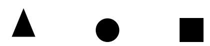


Fig. 1. Representation of a major vertex, a minor vertex, and a vertex with degree $\Delta(T) - 2$.

In this paper we present a complete characterization of the $L(2, 1)$-span of trees up to twenty vertices. We provide a list of forbidden subtrees whose presence will imply that the tree has span $\Delta(T) + 2$. For $\Delta(T) \in \{3, 4\}$, we use the Chang–Kuo algorithm on all 823,065 non-isomorphic such trees on twenty vertices (generated by *nauty*) to obtain part of the result.

2. Forbidden subtree enumeration

In this paper, we refer to a vertex $u \in V(G)$, where $deg(u) = \Delta(G)$, as a *major vertex*. Similarly, we refer to a vertex $v \in V(G)$, where $deg(v) \neq \Delta(G)$, as a *minor vertex*. For convenience, we also refer to the $L(2, 1)$-span of a graph G as $\lambda(G)$ instead of $\lambda_{2,1}(G)$. A tree is *Type I* if $\lambda(T) = \Delta(T) + 1$, and *Type II* if $\lambda(T) = \Delta(T) + 2$.

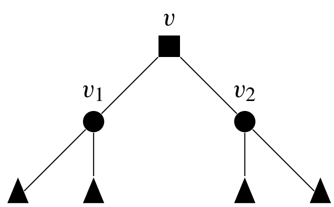
We use the notion of the *critical labels* of an $L(2, 1)$ labeling f to mean the maximum and minimum possible labels in f. For example, given an $L(2, 1)$ labeling of a tree T with $\Delta(T) = 5$ and span $\Delta(T) + 1$, the *lower critical value* is 0 and the *upper critical value* is 6. A *Δ-path segment* is a path P between two major vertices v_i and v_j such that all internal vertices of P are minor vertices. A *forbidden subtree* is a subgraph T' of a tree T such that $\lambda(T') = \Delta(T) + 2$. Note that if such a subtree exists in a tree T, then T is Type II. A *slack vertex* is any minor vertex that belongs to any Δ-path segment. It was shown in [4] that all minor vertices that are not slack vertices in a tree T can be pruned (i.e. removed) without changing $\lambda(T)$. For simplicity, we define a labeling or label assignment as the application of a labeling f on a specified set of vertices. However, for paths $P_n = v_1, v_2, \ldots, v_n$ we denote such label assignments as $\langle f(v_1) f(v_2) \ldots f(v_n)\rangle$.

Fig. 1 shows the visual representation of major vertices, minor vertices, and vertices with degree $\Delta(T) - 2$, respectively. Note that vertices with degree $\Delta(T) - 2$ are also minor vertices, but since they are specifically required in some forbidden subtree structures we will present, we define an explicit visual representation for them here.

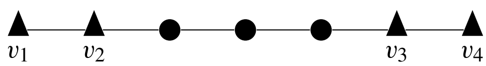
In Theorem 1 we present a complete $L(2, 1)$-span characterization for trees up to twenty vertices.

Theorem 1. *For a tree T on $n \leq 20$ vertices, T is Type II if and only if T exhibits any of the following structural characteristics:*

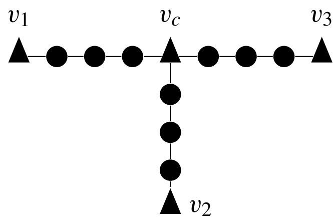
1. T contains an induced P_3 consisting of three major vertices.
2. T contains one minor vertex v that has at least 3 major vertices in $N(v)$.
3. T contains one major vertex v that has $\Delta(T) - 1$ major vertices in $N_2(v)$.
4. T contains one vertex v with degree $\Delta(T) - 2$ that is adjacent to $\Delta(T) - 2$ subtrees $T_1, T_2, \ldots, T_{\Delta(T)-2}$ rooted at vertices $v_1, v_2, \ldots, v_{\Delta(T)-2}$, respectively, where $deg(v_1) = deg(v_2) = \cdots = deg(v_{\Delta(T)-2}) = 3$ and each vertex v_i, $1 \leq i \leq \Delta(T) - 2$, is adjacent to 2 distinct major vertices.



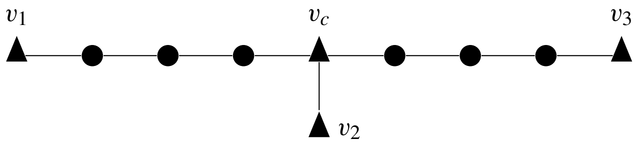
5. $\Delta(T) \leq 3$ and T contains four major vertices v_1, v_2, v_3, v_4 such that $(v_1, v_2), (v_3, v_4) \in E(T)$, and $d(v_2, v_3) = 4$ and $d(v_1, v_4) = 6$.



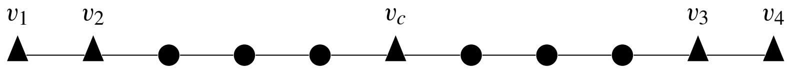
6. $\Delta(T) = 3$ and T contains one major vertex v_c that has three distinct major vertices at distance 4. In other words, v_c is the endpoint for three separate Δ-path segments of length 4.



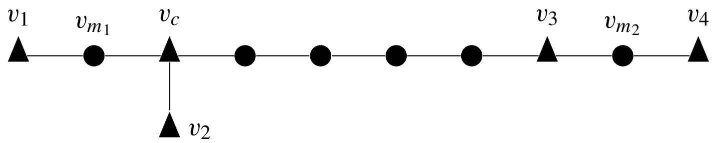
7. $\Delta(T) = 3$ and T contains one major vertex v_c that has two distinct major vertices v_1, and v_3 at distance 4 and one major vertex v_2 at distance 1 such that v_2 does not belong to the $v_c v_1$ or the $v_c v_3$ path.



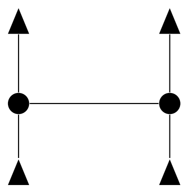
8. $\Delta(T) = 3$ and T contains one major vertex v_c that has two distinct major vertices v_2 and v_3 at distance 4, which each have one distinct major vertex v_1 and v_4, respectively, at distance 1 such that $d(v_c, v_1) = d(v_c, v_4) = 5$.



9. $\Delta(T) = 3$ and T contains one major vertex v_c where $d(v_c, v_2) = 1$, $d(v_c, v_1) = 2$, $d(v_c, v_3) = 5$, and $d(v_c, v_4) = 7$ for distinct major vertices v_1, v_2, v_3, and v_4, as below.



10. $\Delta(T) = 4$ and T contains two Δ-path segments of length 2, that are connected by an edge between their internal minor vertices.



One proof of Theorem 1 is to enumerate all trees, determine the spans, check for the substructures and verify that it matches with the list in Theorem 1. We use a modified approach here. In Section 3 we prove that any tree, regardless of order, with one of the substructures listed in Theorem 1 is Type II. In Section 4, we provide the proof of the sufficient part of Theorem 1, with the help of a computer program in some cases. However, we know that there are Type II trees of order greater than 20 that do not have one of the ten substructures listed in Theorem 1.

3. Proof of necessary part of Theorem 1

We start with Theorem 2 mentioned in [2]. For completeness, we provide the proof of Theorem 2 here.

Theorem 2. *Let T^* be a Type I tree. Then for any labeling f of T^* with span $\Delta(T^*) + 1$, $f(v_m) \in \{0, \Delta(T^*) + 1\}$ for all major vertices $v_m \in V(T^*)$.*

Proof. This comes from a straight forward application of the pigeonhole principle. If $f(v_m) \notin \{0, \Delta(T^*) + 1\}$ for any major vertex $v_m \in V(T^*)$, then the label of v_m removes three possible labels from the label set $S = \{0, \dots, \Delta(T^*) + 1\}$. The resulting label set S' has cardinality $\Delta(T^*) - 1$. However, as $|N(v)| = \Delta(T^*)$, it is not possible to assign unique labels to all of the vertices in $N(v)$. $\square$

Using similar arguments as in the case of the proof of Theorem 2, we get the following.

Corollary 3. *If T^* has at most 2 major vertices, then T^* is a Type I tree.*

Lemma 4. *If a tree T contains one of the ten subtrees listed in Theorem 1 then T is Type II.*

Proof. (1) Since each major vertex v must receive either an upper or lower critical label in an $L(2, 1)$ labeling of span $\Delta(T) + 1$, we can only assign critical labels to two out of the three major vertices in $\{v_1, v_2, v_3\}$. This is because each major vertex v in $\{v_1, v_2, v_3\}$ is adjacent to either two other major vertices or adjacent to one major vertex and at a distance of 2 from the other major vertex.

(2) Since each major vertex v must receive an upper or lower critical label in an $L(2, 1)$ labeling of span $\Delta(T) + 1$, we can assign two distinct critical labels to only two of the vertices in $\{v_1, v_2, v_3\}$. Assigning a critical label to the third unlabeled vertex in $\{v_1, v_2, v_3\}$ will violate the $L(2, 1)$-labeling condition that vertices at distance 2 must have distinct labels.

(3) Let v be a major vertex with $\Delta(T) - 1$ major vertex neighbors $v_1, v_2, v_3, \ldots, v_k$ at distance 2. Assume, for the sake of contradiction, that a $\Delta(T) + 1$ labeling exists. Therefore, we can assign the lower critical value to v and the upper critical value to $v_1, v_2, v_3, \ldots,$ and v_k. Since there are a total of $\Delta(T) + 2$ possible label values to choose from, this labeling leaves $\Delta(T) - 2$ labels left to choose from for vertices in $N(v)$. However, since $|N(v)| = \Delta(T) - 1$ and there are only $\Delta(T) - 2$ labels to use, by the pigeonhole principle it is not possible to uniquely label all of the vertices in $N(v)$ without duplicating one.

(4) Assume we have a labeling f of T with span $\Delta(T) + 1$. Let $v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, \ldots, v_{\Delta(T)-2,1}, v_{\Delta(T)-2,2}$ be the major vertices adjacent to vertices $v_1, v_2, \ldots, v_{\Delta(T)-2}$. Without loss of generality, each pair of vertices $v_{i,1}$ and $v_{i,2}$, $1 \leq i \leq \Delta(T) - 2$, must receive the labels 0 and $\Delta(T) + 1$, respectively, because they belong to the same subtree T_i that is rooted at vertex v_i. Since $d(v_i, v_j) = 2$ for all vertices v_i and v_j, it must be true that $f(v_i) \neq f(v_j)$ and $f(v_i), f(v_j) \notin \{0, 1, \Delta(T), \Delta(T) + 1\}$. Since there are $\Delta(T) - 2$ such root vertices and $\Delta(T) + 2 - 4 = \Delta(T) - 2$ possible labels, we can assign each vertex v_i, $1 \leq i \leq \Delta(T) - 2$, a unique label. However, at this point, we cannot label v from the set $\{0, 1, \ldots, \Delta(T), \Delta(T) + 1\}$.

(5) If $\Delta(T) \leq 2$, then T is a path on at least 7 vertices and $\lambda(P_7) = 3$. Assume $\Delta(T) = 3$. Since the pairs of major vertices in T are directly adjacent to one another, they must receive the labels 0 and 4, respectively. Without loss of generality, let $f(v_1) = 4$, $f(v_2) = 0$, $f(v_3) \in \{0, 4\}$, and $f(v_4) \in \{0, 4\}$. If $f(v_3) = 0$, then we know that $f(v_4) = 4$. With this partial labeling, the only possible label scheme for the Δ-path segment between v_2 and v_3 is $\langle 03240 \rangle$. However, this conflicts with the label for v_4. Now, consider the alternate major labeling scheme where $f(v_3) = 4$ and $f(v_4) = 0$. With this new partial labeling there is no labeling f for the vertices of the Δ-path segment between v_2 and v_3 with a span of $\Delta(T) + 1$.

(6) Let f be an $L(2, 1)$ labeling of T with span $\Delta(T) + 1 = 4$. Note that, in this case, there does not exist a labeling f of span $\Delta(T) + 1$ for any Δ-path segment of length 4 when the inner path vertices start at 2. This is because, starting at the middle major vertex, the only possible labels for this Δ-path segment belong to the set $\{\langle 02402 \rangle, \langle 02403 \rangle, \langle 02413 \rangle, \langle 42042 \rangle, \langle 42041 \rangle, \langle 42031 \rangle\}$. However, in each of these cases, a major vertex will not have a critical value, which contradicts Theorem 2. So, starting at the middle major vertex, the label assignment for any Δ-path segment of length 4 must be in the set $\{\langle 03140 \rangle, \langle 04130 \rangle, \langle 04204 \rangle, \langle 41304 \rangle, \langle 40314 \rangle, \langle 40240 \rangle\}$. Since the first label in each of these assignments is the label of the middle major vertex, the three Δ-path segments must receive either the first three assignments in this set, or the last three assignments. In either case, two minor vertices adjacent to v_c will receive the same label.

(7) Let f be an $L(2, 1)$ labeling of T with span $\Delta(T) + 1 = 4$. Since there is one major vertex immediately adjacent to v_c, we know it must receive a critical label. Thus, one of the other Δ-path segments must start with a label of 2, and by the same argument in the proof of (6), we conclude that $\lambda(T) = \Delta(T) + 2 = 5$.

(8) Let f be an $L(2, 1)$ labeling of T with span $\Delta(T) + 1 = 4$. Since there are two pairs of adjacent major vertices, we assign them labels from the set $\{0, 4\}$. If we fix the labels for one pair of major vertices and force the labels towards the center of T, we see the only possible label assignments for the first half of the subtree T, composed of the path $P = v1, v2, \ldots, v_c$, belong to the set $\{\langle 041304 \rangle, \langle 403140 \rangle\}$. Now, if we continue forcing label assignments towards the second pair of adjacent major vertices $\{v_3, v_4\}$, we see that all possible label assignments yield a critical label on the minor vertex immediately adjacent to v_3. This critical label will conflict with the labels for v_3 and v_4.

(9) Let f be an $L(2, 1)$ labeling of T with span $\Delta(T)+1 = 4$. Since v_c, v_1, and v_2 are all major vertices, they must be given critical labels, which implies that $f(v_{m_1}) = 2$, where v_{m_1} is the minor vertex adjacent to v_2 and v_c. Now, with this partial labeling we continue to choose labels towards v_3 and v_4 such that f still has a span of $\Delta(T) + 1$. In particular, the only possible labelings for the path from v_c to v_3 belong to the set $\{\langle 031420\rangle, \langle 413024\rangle\}$. However, since v_3 and v_4 are also major vertices, we know that $f(v_{m_2}) = 2$, where v_{m_2} is the common neighbor between v_3 and v_4, which conflicts with the other minor vertex adjacent to v_3.

(10) Let f be an $L(2, 1)$ labeling of T with span $\Delta(T) + 1 = 5$. By the definition of a Δ-path segment, the endpoints of both segments must receive the labels of 0 and 5, respectively. Therefore, the only remaining label choices for the minor vertices are $\{2, 3\}$, and since they are adjacent this contradicts the assumption that the span of f is $\Delta(T) + 1$. □

4. Proof of sufficient part in Theorem 1

To prove the sufficient criteria in Theorem 1, we consider trees up to twenty vertices on a case-by-case basis using their maximum degree and show that if a tree does not contain the forbidden subtrees in Theorem 1 then it is a Type I tree.

We first recall the concept of a *slack vertex*, defined in Section 2, which is a minor vertex in a tree that belongs to a Δ-path segment. It was shown in [4] that all minor vertices that are not slack vertices in a tree T can be pruned (i.e. removed) without changing $\lambda(T)$. We also make the observation that for a tree T of order n such that $\lambda(T) = \Delta(T) + 1$, an upper bound on the number of major vertices $M(n)$ is

$$M(n) = \left\lfloor \frac{n-2}{\Delta(T)-1} \right\rfloor, \tag{1}$$

which comes from the situation where major vertices form an induced subgraph isomorphic to $P_{M(n)}$. From this value, we can see that for a tree T of order n with a fixed $\Delta(T) = d$ and number of major vertices $M(n)$, the maximum number of slack vertices that can be inserted in Δ-path segments of T is

$$S(d, n) = n - 2d - (M(n) - 2)(d - 1). \tag{2}$$

The construction technique works by examining each possible maximum degree $5 \leq d \leq 9$ and attempting to construct all trees T of order $n = 20$ with $\Delta(T) = d$. For $d = 3, 4$ the number of possible constructions becomes quite large, so for these cases we follow a different approach. Specifically, we rely on *nauty* [5] to generate all non-isomorphic trees on 20 vertices. From this set of 823065 trees, we separate Type I and Type II trees. For each Type II tree, we then programmatically check to see whether or not the forbidden subtrees listed in Theorem 1 are present. Our results are below.

Suppose T does not contain any of the trees listed in Theorem 1 as subtrees. Note that, by Corollary 3, if T has at most two major vertices, then T is Type I. So we assume that T has at least three major vertices.

Case 1: $\Delta(T) \geq 8$

By Eq. (1), we know that the maximum number of major vertices is $\lfloor(20-2)/(8-1)\rfloor = \lfloor(20-2)/(9-1)\rfloor = 2$. Thus, we know that it is impossible to have more than 2 major vertices, and so T is Type I.

Case 2: $\Delta(T) = 7$

By Eq. (1), we know that the maximum number of major vertices is $\lfloor(20 - 2)/(7 - 1)\rfloor = 3$, but this case only occurs when the 3 major vertices form an induced P_3. Since this corresponds to the first forbidden subtree, we know this cannot occur, and thus there can be at most two major vertices, and so T is Type I.

Case 3: $\Delta(T) = 6$

By Eq. (1), we know that the maximum number of major vertices is$\lfloor(20 - 2)/(6 - 1)\rfloor = 3$. By Eq. (2), we know that there are a maximum of 3 slack vertices possible. Thus, considering trees that have been pruned, we can enumerate all possible structures with 3 major vertices and up to 3 slack variables that do not contain any of the forbidden subtrees in Section 3. It is easy to see that the remaining structures, which are shown in Fig. 2, have an $L(2, 1)$-span of $\Delta(T) + 1$.

Case 4: $\Delta(T) = 5$

By Eq. (1), we know that the maximum number of major vertices is $\lfloor(20-2)/(6-1)\rfloor = 4$ and by Eq. (2) we know that there are a maximum of two slack vertices possible with four major vertices. If we relax the major vertex count

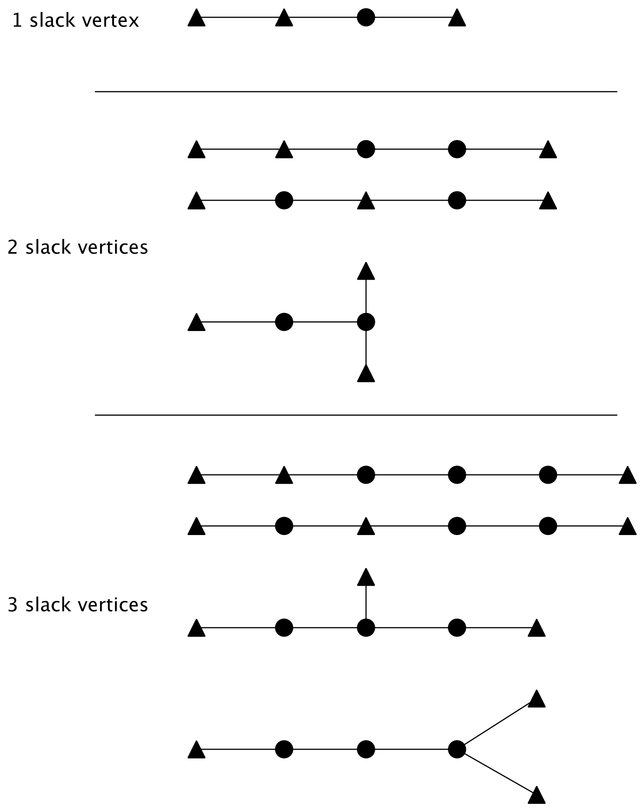


Fig. 2. All graph structures with 3 major vertices and up to 3 slack vertices, and $\Delta(T) = 6$.

down to 3, we can now construct trees with up to 7 slack variables. Thus, considering trees that have been pruned, we can enumerate all possible structures with 3 and 4 major vertices and up to 3 and 7 slack variables, respectively, using the same approach as in the $\Delta(T) = 6$ case. It is easy to verify that the resulting trees are Type I.

Case 5: $\Delta(T) \in \{3, 4\}$

The approach used for $\Delta(T) = 5$ case does not work for this case as the number of possible structures become significantly large. All trees T up to twenty vertices and $\Delta(T) = 3$ and $\Delta(T) = 4$ were exhaustively checked with a software implementation of the Chang–Kuo algorithm on trees generated by *nauty*. No Type II trees were found where T did not contain one of the forbidden subtrees.

5. Conclusion

In this paper we presented a complete $L(2, 1)$-span characterization for trees up to twenty vertices. This characterization is an indication of the underlying complexity of the relationship of a tree's structure and $L(2, 1)$-span. For trees with smaller maximum degrees, this technique depended on the enumeration of all trees on $n \leq 20$ vertices. With the computational facilities available of us, it took more than a week to do our exhaustive search, which suggests that doing the same approach for higher order trees is infeasible. Also, there exist more unique forbidden subtrees for higher order trees; for example, we were able to find a new forbidden subtree for $n = 23$.

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